2019-Sep-19 (lecture)

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Previously: We proved that we can find solutions to first-order autonomous ODEs $\dot{x} = f(x)$, demonstrating **existence**.

More than that, if $f(x_0) \neq 0$, we could explicitly write down the only local solution by defining

$$F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}, \text{ and taking } \phi(t) = F^{-1}(t).$$

That makes the solution **locally unique**.

If the interval of validity $(x_1, x_2) = \mathbb{R}$, then the solution is **globally unique**.

Around some x_0 s.t. $f(x_0) = 0$, however, we could extend the solution in a **non-unique** way, only if

$$\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| < \infty$$

(Hint: Teschl 1.10 bonus) When $\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| = \infty$ for all $x_0 \in \mathbb{R}$ s.t. $f(x_0) = 0$, we thus again have a **globally unique** solution.

Can we say anything about general ODEs?

We are going to prove **local existence** and **uniqueness** for solutions to general ODEs.

Note: We are not **solving** them, just proving stuff about the solutions.

Intuition:

Fixed point: Given an arbitrary mapping $f: X \to X$, any point x where $f(x) = x$.

Ex.

$X = \mathbb{R}$:

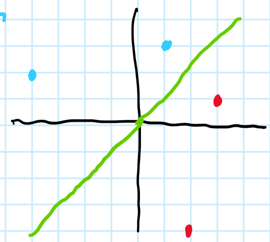
$f(x) = x^2$. Fixed points 0 and 1

$f(x) = \frac{x}{2}$. Fixed points 0

$f(x) = x+1$. No fixed points

$X = \mathbb{R}^2$:

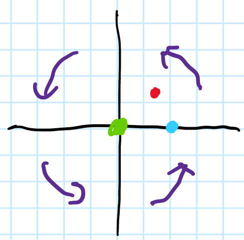
$f(x) = Ax$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $x \in \mathbb{R}^2$



Fixed points $x_1 = x_2$

$f(x) = Ax$ $\quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotation.

$\theta = 45°$



Fixed point $(0,0)$

$X = C^\infty(\mathbb{R})$ – i.e. let's consider the space of infinitely-differentiable functions on the real line.

Let our mapping be $\frac{d}{dx}$.

What are the fixed points?

$$\frac{d}{dx} f(x) = f(x)$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} 0 = 0$$

$$\frac{d}{dx}(Ce^x) = Ce^x.$$

What about the mapping $K(f)$ defined by

$$K(f)(x) = 1 + \int_0^x f(y)\,dy\ ?$$

$T(f)(x) = 1 + \int_0^x f(y)\,dy$.

i.e. need $f(x) = 1 + \int_0^x f(y)\,dy$ (Integral Equation)

Try $f(x) = e^x$. $e^x = 1 + \int_0^x e^y\,dy = 1 + e^y\Big|_0^x = 1 + e^x - e^0 = e^x$.

So, we can find fixed points for all kinds of mappings.

Contractions: A mapping that makes everything get closer

Ex. $f(x) = \frac{x}{2}$. $|f(x) - f(y)| = \frac{1}{2}|x - y| \le K|x - y|$, $K \in [0,1)$

$f(x) = x^2$ not contraction

$f(x) = \sqrt{x}$ contraction

$f(x) = 1000 + \frac{x}{2}$

$f(x) = Ax$, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ not contraction

$f(x) = Ax$, $A = \begin{bmatrix} 0.5\cos\theta & -0.5\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ contraction

$\frac{d}{dx}$, $\int$ N/A

We haven't defined distance yet for functions.

Notice: When things get closer, there's "always" a single unique fixed point.

We will show that (in a variety of settings), given a contraction, there exists a unique fixed point.

TODO:

- Define distance for functions
- Show that a certain mapping on functions is a contraction
- Show that mapping's fixed pt is the solution to an ODE.

More rigorously:

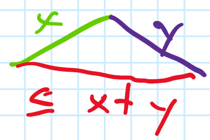
Let X be a real vector space. A norm (distance) on X is a map $\|\cdot\| : X \to [0, \infty)$ s.t.

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \neq 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$

(triangle inequality)



Together, $(X, \|\cdot\|)$ is a normed vector space.

A sequence of vectors x_n converges to x if

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

e.g. $x_n = \left(\frac{1}{10^n}, 1 - \frac{1}{10^n} \right) = \{(0.1, 0.9), (0.01, 0.99), \ldots\}$

converges to $(0, 1)$

A mapping $F : X \to Y$ between normed vector spaces is called continuous if $x_n \to x$ implies that $F(x_n) \to F(x)$.

Note: (Teschl 2.2) The norm, vector addition, and multiplication by scalars are continuous.

When do sequences converge? When they close to one another

A sequence $x_1, x_2, \ldots$ is a Cauchy sequence if for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $m, n > N$, $\|x_m - x_n\| < \varepsilon$.

A space is called complete if every Cauchy sequence has a limit.

A complete normed space is called a Banach space

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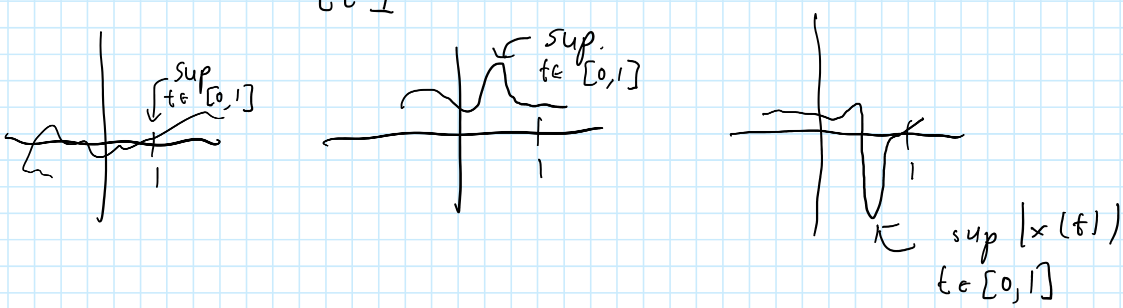
A complete normed space is called a Banach space

ex. $\mathbb{R}^n$, $\mathbb{C}^n$ are Banach spaces (Euclidean norm)

But what about function spaces?

Let $I \subseteq \mathbb{R}$ be a closed finite interval on the real line.
(compact) e.g. $I = [0,1]$

Define $\|x\| = \sup_{t \in I} |x(t)|$.



Consider $C(I)$, the space of continuous functions on I.

Is this a metric? Check all three properties.